Solutions



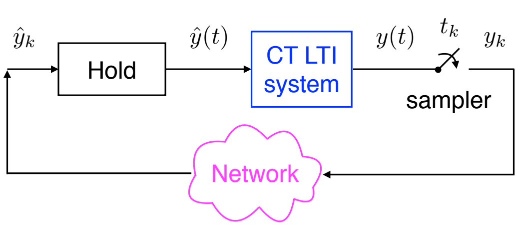
**Problem 1.** /21

When there are multiple choices in the following, select all statements that are true.

1. Consider the discrete-time switched system $x(k+1) = A_{\sigma(k)}x(k)$, $x(k) \in \mathbb{R}^3$ where $\sigma(k) \in \{1, 2\}$.
   - ☒ If $\exists P \in \mathbb{R}^{3\times3}$, $P = P^T > 0$ such that
   $$A_1^T PA_1 - P < 0 \quad \text{and} \quad A_2^T PA_2 - P < 0$$
   then the system is exponentially stable.
   - ◯ If the system is exponentially stable, then there is $P \in \mathbb{R}^{3\times3}$, $P = P^T > 0$ such that
   $$A_1^T PA_1 - P < 0 \quad \text{and} \quad A_2^T PA_2 - P < 0$$
   - ◯ If $\mathrm{Spec}(A_1) = \{1+j, 1-j, 0\}$, then the system cannot be exponentially stable.

2. Consider the LTI system $x(k+1) = Ax(k)$, $x(k) \in \mathbb{R}^n$.
   - ☒ If there is a matrix $P = P^T > 0$ such that $A^T PA - P = -I$, then, for all $x(0) \in \mathbb{R}^n$, $x(k)$ is bounded.
   - ☒ If, for all $x(0) \in \mathbb{R}^n$, $\lim_{k\to+\infty} x(k) = 0$, then $\exists \alpha > 0, \rho \in [0, 1)$ such that $\|x(k)\| \le \alpha\rho^k\|x(0)\|$, $\forall x(0) \in \mathbb{R}^n$.
   - ◯ If $x(k)$ is bounded for all $x(0) \in \mathbb{R}^n$, then there is a matrix $P = P^T > 0$ such that $A^T PA - P = -I$.

3. Consider the NCS given in the figure, characterized by packet dropouts, network-induced delay $\tau = 0$, and constant sampling period $T > 0$.




CT LTI system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

As seen in the lectures, the discrete-time NCS model is

$$z_{k+1} = \psi_{\theta_k} z_k, \quad z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}, \quad \psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

where $\Gamma(s) = \int_0^s e^{At}dt$ and $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \dots$. Assume that the asymptotic packet dropout rate $r \in [0, 1]$ exists.

   - ☒ If $r = 0.5$ and there are $P = P^T > 0$ and $\alpha, \alpha_0, \alpha_1$ such that
   $$\sqrt{\alpha_o\alpha_1} > \alpha > 1, \quad \psi_0^T P\psi_0 \le \alpha_0^{-2}P, \quad \psi_1^T P\psi_1 \le \alpha_1^{-2}P$$
   then the NCS is exponentially stable.

☒ If $r = 1$ the NCS cannot be asymptotically stable.

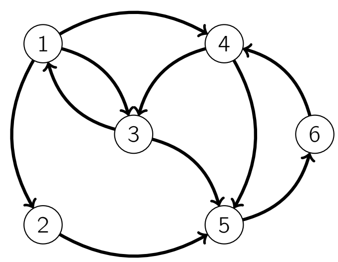
☒ If there is $P = P^T > 0$ such that

$$\psi_0^T P \psi_0 \leq e^3 P, \quad \psi_1^T P \psi_1 \leq e^{-\frac{1}{2}} P$$

then the NCS is exponentially stable for all $r < \frac{1}{7}$.

$2 \leq \frac{1}{1-\frac{3}{-\frac{1}{2}}}$ $\frac{1}{1+6} = \frac{1}{7}$

4. Let $A$ be the binary adjacency matrix of the digraph in the figure.




○ $A$ is reducible.

☒ The element $(4, 2)$ of $A^5$ is nonzero.

☒ The element $(4, 2)$ of $\sum_{k=0}^{4} A^k$ is nonzero.

5. Let $G$ be a weighted graph with $n > 2$ nodes and let $L$ be its Laplacian matrix.

☒ If $G$ is directed and a node $v$ is a sink, then $L_{v,v} = 0$.

○ If $G$ is undirected and connected, then the eigenvalue $\lambda = 0$ of $L$ has algebraic multiplicity 2.

☒ If $G$ is directed and contains a globally reachable node, then all and only equilibria of $\dot{x} = -Lx$ are the states $\bar{x} = \alpha \mathbb{1}_n$, $\alpha \in \mathbb{R}$.

6. Let $T = (V, E)$ be an undirected graph with $V = \{1, 2, \ldots, 10\}$. Assume $T$ is a tree.

○ $T$ has 10 edges.

☒ Removing an edge from $T$ makes $T$ disconnected.

☒ $T$ is connected.

7. Assume that the digraph $G$ with vertex set $V = \{1, 2, \ldots, 5\}$ is weakly connected and has a strongly connected component induced by the vertices $\{1, 2, 3\}$. Then,

○ $G$ contains the cycle $(1, 2)$, $(2, 3)$, $(3, 1)$.

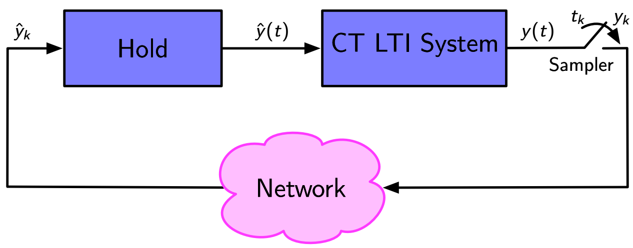
○ The subgraph induced by the set of vertices $\{1, 2, 4\}$ is strongly connected.

☒ The condensation graph of $G$ has at least two nodes.

**Problem 2.**

/24

Consider the NCS in the figure below




where the LTI system is the first-order model

$$\begin{cases} \dot{x} = x + 2u \\ y = cx \end{cases}.$$

1. Assume that the sampling time $T = \log(4)$ is constant and the network induced delay $\tau < T$ is constant but unknown. Compute the inequalities characterizing all values of $c$ and $\tau$ for which the NCS is asymptotically stable. Find then all values of $c$, if any, guaranteeing asymptotic stability for $\tau = \log(2)$.

   **Hint:** Recall the Jury's criterion: the roots of $\phi(\lambda) = \lambda^2 + \alpha\lambda + \beta$ verify $|\lambda| < 1$ if and only if
$$\begin{aligned} \beta &> -\alpha - 1 \\ \beta &> \alpha - 1 \\ \beta &< 1 \end{aligned}.$$

The discrete-time NCS model with state $z_k^T = [x(t_k), \hat{y}_{k-1}]$ is given by

$$z_{k+1} = \Psi(T, \tau) z_k$$

$$\Psi(T, \tau) = \begin{bmatrix} e^{AT} + \Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B \\ C & 0 \end{bmatrix}$$

where $A = 1$, $B = 2$

Computations

$$e^{AT} = e^{\log 4} = 4$$

$$e^{A(T-\tau)} = e^{AT} e^{-A\tau} = 4e^{-\tau}$$

$$\Gamma(\tau) = e^{\tau} - 1$$

Hence

$$\Psi(T,\tau) = \begin{bmatrix} \underbrace{4+8c\,e^{-\tau}-2c}_{\Psi_{11}} & \underbrace{2\cdot 4 e^{-\tau}(e^{\tau}-1)}_{\Psi_{12} = 8-8e^{-\tau}} \\ c & 0 \end{bmatrix}$$

The characteristic polynomial of $\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ c & 0 \end{bmatrix}$ is

$$\det\left(\lambda I - \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ c & 0 \end{bmatrix}\right) = \lambda^2 - \Psi_{11}\lambda - c\,\Psi_{12}$$

By applying Jury's criterion, one has

$$\begin{cases} -c\,\Psi_{12} > \Psi_{11} - 1 \\ -c\,\Psi_{12} > -\Psi_{11} - 1 \\ -c\,\Psi_{12} < 1 \end{cases} \longrightarrow \begin{cases} c\,\Psi_{12} < 1 - \Psi_{11} \\ c\,\Psi_{12} < 1 + \Psi_{11} \\ c\,\Psi_{12} > -1 \end{cases}$$

$$\begin{cases} 8c(1-e^{-\tau}) < 1-4-8ce^{-\tau}+2c = -3-8ce^{-\tau}+2c & (*) \\ 8c(1-e^{-\tau}) < 1+4+8ce^{-\tau}-2c = 5+8ce^{-\tau}-2c & (**) \\ 8c(1-e^{-\tau}) > -1 & (***) \end{cases}$$

For $\tau = \log(2)$, $e^{-\tau} = \dfrac{1}{e^{\log(2)}} = \dfrac{1}{2}$

$(*) \rightarrow \underbrace{8c(1-\tfrac{1}{2})} < -3 - \tfrac{8}{2}c + 2c \rightarrow 4c < -3-2c \rightarrow 6c < -3 \rightarrow c < -\tfrac{1}{2}$

$(**) \rightarrow 4c < 5 + 4c - 2c \longrightarrow 4c < 5 \longrightarrow c < \tfrac{5}{2}$

$(***) \rightarrow 4c > -1 \longrightarrow c > -\tfrac{1}{4}$ No value of $c$ verifies $c < -\tfrac{1}{2}$ and $c > -\tfrac{1}{4}$

2. Assume now that $T = \log(4)$, $c = 1$ but the delay $\tau_k$ is time-varying in the interval $[0.1, 0.2]$. Give sufficient conditions for certifying the exponential stability of the NCS through a quadratic Lyapunov function.

The NCS is exponentially stable if $\exists P = P^T > 0$, $P \in \mathbb{R}^{2\times 2}$ and $\nu > 0$ such that

$$\Psi(T, \tau)^T P \Psi(T, \tau) - P \leq -\nu I \quad \forall \tau \in [0.1, 0.2]$$

The lyapunov function certifying exponential stability is

$$V(x_k) = x_k^T P x_k$$

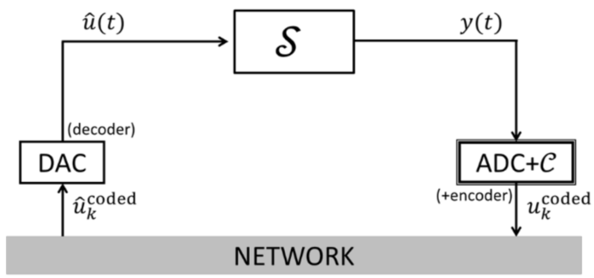
**Problem 3.**

/15

Consider the first-order continuous-time system $\mathcal{S}$ with dynamics

$$\dot{x} = \frac{1}{2}x + 2u$$

along with the NCS shown in the figure below, where the network is ideal. Assume the system is controlled in a sample-and-hold fashion, that is $u(t) = u_k$ for $t \in [kT, (k+1)T)$, $k \geq 0$. Moreover, assume that $u_k \in \mathcal{U}$, where $\mathcal{U}$ contains $2^{N_B}$ values.



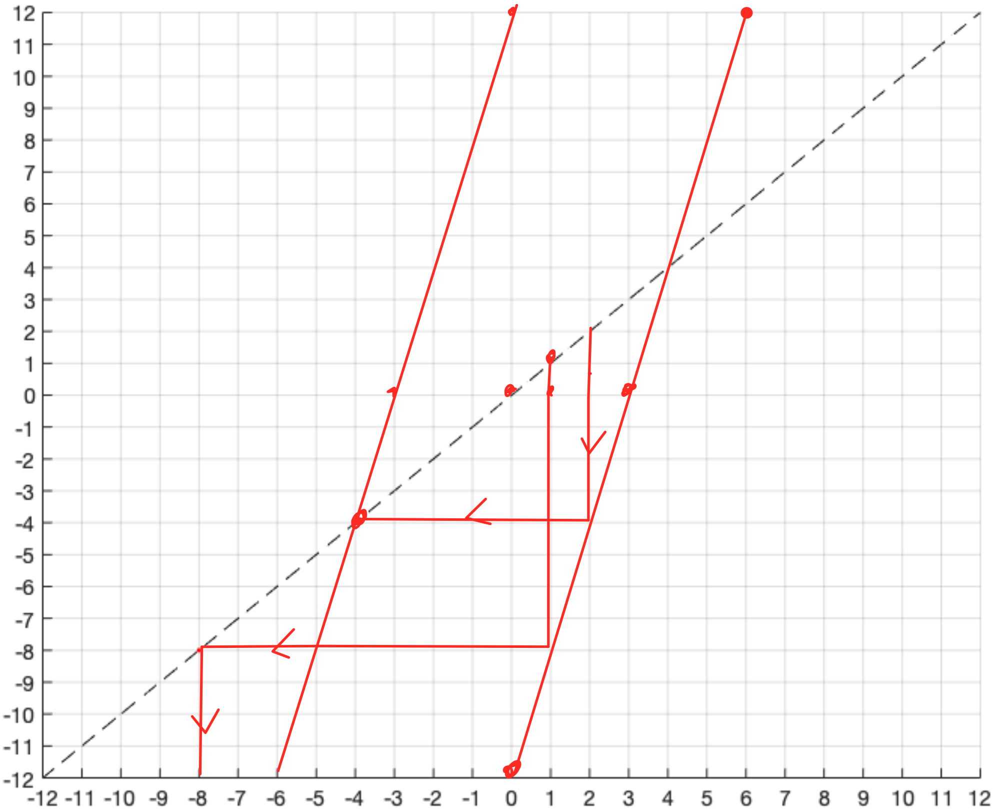

1. Compute the minimun rate $\frac{N_B}{T}$ required to make the system boundable. For $T = 2\log(4) = 2.7726$ s, compute the minimum number of bits $N_B$ to be transmitted within each sampling interval to make the system boundable.

$a = \frac{1}{2}$ $b = 2$

$\frac{N_B}{T} \geq a \cdot \log_2 e = a \cdot 1.4427 = 0.7214$

$N_B \geq 0.7214 \cdot T = 1.999$ and, being $N_B$ an integer, one has $N_B = 2$

2. Set $T = 2\log(4)$ s, and assume that $N_B = 1$. Using the control law $u_k = -\text{sign}(x_k)$, show the properties of the state trajectories using the graphical method. Plot the results in the graduated figure below, starting at least from $x_0 = 1$ and $x_0 = 2$.




• DT system:

$$x^+ = f x + g u \qquad f = e^{aT} \qquad g = -\frac{b}{a}\left(1 - e^{aT}\right)$$

$\downarrow a = \frac{1}{2}$ $\qquad \downarrow b = 2$

$$f = 4 \qquad g = -\frac{2}{\frac{1}{2}}(1-4) = +4\cdot 3 = 12$$

• Closed-loop dynamics:

$$x_{k+1} = \begin{cases} f x_k - g & \text{if } x_k \geq 0 \\ f x_k + g & \text{if } x_k < 0 \end{cases}$$

$$x_{k+1} = \begin{cases} 4 x_k - 12 & \text{if } x_k \geq 0 \\ 4 x_k + 12 & \text{if } x_k < 0 \end{cases}$$

. equilibria $x_k = 4$ and $x_k = -4$.

. For almost all $x_0$, trajectories are diverging

The control law is given by

$$s(x) = \begin{cases} u_{max} = 1 & \text{if } x \leq \bar{x}_1 \Rightarrow x \leq -2 \\ \bar{u}_1 = \frac{1}{3} & \text{if } \bar{x}_1 < x \leq \bar{x}_2 \rightarrow -2 < x \leq 0 \\ \bar{u}_2 = -\frac{1}{3} & \text{if } \bar{x}_2 < x \leq \bar{x}_3 \rightarrow 0 < x \leq 2 \\ \bar{u}_3 = -1 & \text{if } \bar{x}_3 < x \rightarrow x > 2 \end{cases}$$

and the positively invariant set is

$$\mathcal{M} = [\bar{x}_0, \bar{x}_N] = [-4, 4]$$

3. Set $T = 2\log(4)$ s, $N_B = 2$ and $\mathcal{U} = \cancel{\{-1, 1\}}$ $[-1, 1]$. Compute the control law for guaranteeing boundability when one operates at the rate limits. Compute the corresponding positively invariant set.

The control law at the rate limits is given by

$$N = 2^{N_B} = 4$$

$$\bar{u}_i = u_{max} + \frac{i}{4-1}(u_{min} - u_{max}) = 1 - \frac{2i}{3} \rightarrow \begin{cases} \bar{u}_0 = 1 \\ \bar{u}_1 = 1 - \frac{2}{3} = \frac{1}{3} \\ \bar{u}_2 = 1 - \frac{4}{3} = -\frac{1}{3} \\ \bar{u}_3 = 1 - \frac{6}{3} = -1 \end{cases}$$

Set also for $i = 0, 1, \dots, 4$

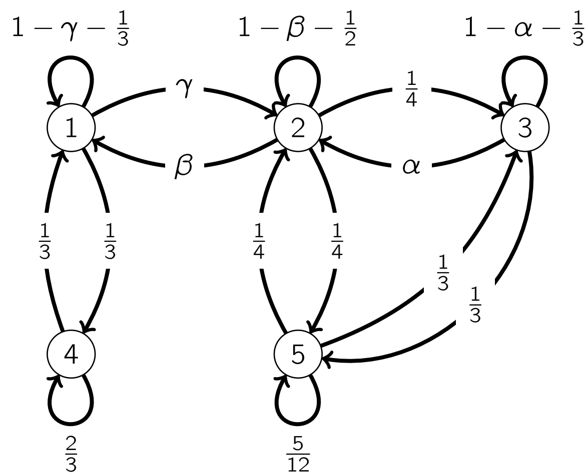
$$\bar{x}_i = -\frac{b}{a} u_{max} + \frac{b}{a}(u_{max} - u_{min})\frac{i}{N} = -4 + \cancel{4} \cdot \frac{2 \cdot i}{\cancel{4}} = -4 + 2i$$

$\hookrightarrow$ $\bar{x}_0 = -4$ $\bar{x}_1 = -2$ $\bar{x}_2 = 0$ $\bar{x}_3 = 2$ $\bar{x}_4 = 4$

**Problem 4.**

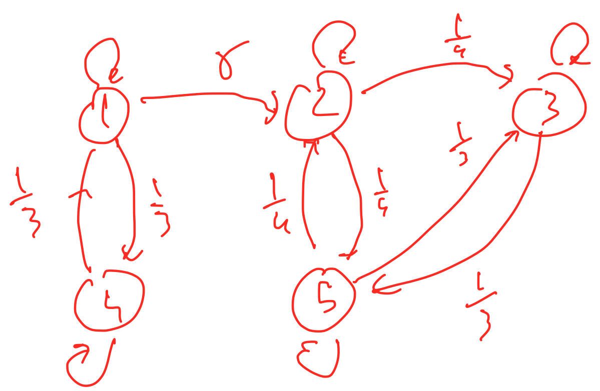
/20

Let $A$ be the adjacency matrix of the digraph $G$ in the figure below and consider the discrete-time system $x(k+1) = Ax(k)$.




1. Set $\alpha = \beta = 0$. Compute all values of $\gamma > 0$ such that $A$ is stochastic and the system reaches a consensus state, as $k \to +\infty$. Compute also the consensus value.

$$A = \begin{bmatrix} 1-\gamma-\frac{1}{3} & \gamma & 0 & \frac{1}{3} & 0 \\ \beta & 1-\beta-\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \alpha & 1-\alpha-\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{3} & 0 & \frac{5}{12} \end{bmatrix} \xrightarrow{\alpha,\beta=0} \begin{bmatrix} 1-\gamma-\frac{1}{3} & \gamma & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{3} & 0 & \frac{5}{12} \end{bmatrix}$$




$GRNs = \{2, 3, 5\}$

$A \geq 0$ if $\begin{cases} \gamma > 0 \\ 1-\gamma-\frac{1}{3} > 0 \end{cases}$

$\downarrow$

$\gamma > 0, \ \frac{2}{3} > \gamma$

A is stochastic and the subgraph induced by GRN is aperiodic

$\Rightarrow x \to w^T x(0) \mathbb{1}_5$ , $w = [0, w_2 \ w_3 \ 0 \ w_5]$

Computation of $w$

$$\begin{bmatrix} 0 & w_2 & w_3 & 0 & w_5 \end{bmatrix} \begin{bmatrix} 1-\gamma-\frac{1}{3} & \gamma & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{3} & 0 & \frac{5}{12} \end{bmatrix} = \begin{bmatrix} 0 & w_2 & w_3 & 0 & w_5 \end{bmatrix}$$

$$\begin{cases} 0 = 0 \\ \frac{1}{2} w_2 + \frac{1}{4} w_5 = w_2 \quad \rightarrow \quad \frac{1}{4} w_5 = \frac{1}{2} w_2 \rightarrow w_5 = 2 w_2 \\ \frac{1}{4} w_2 + \frac{2}{3} w_3 + \frac{1}{3} w_5 = w_3 \quad \rightarrow \quad \frac{1}{4} w_2 + \frac{1}{3}(2 w_2) = \frac{1}{3} w_3 \\ 0 = 0 \\ \frac{1}{4} w_2 + \frac{1}{3} w_3 + \frac{5}{12} w_5 = w_5 \quad \rightarrow \quad \frac{1}{4} w_2 + \frac{1}{3} w_3 = \frac{7}{12} w_5 \end{cases}$$

From the second equation (keeping $w_2$ as free parameter)

$$\frac{1}{3} w_3 = \underbrace{\left(\frac{1}{4} + \frac{2}{3}\right)}_{\frac{3+8}{12}} w_2 \rightarrow w_3 = 3 \, \frac{11}{12} w_2 = \frac{11}{4} w_2$$

From the third equation we obtain the same, as expected

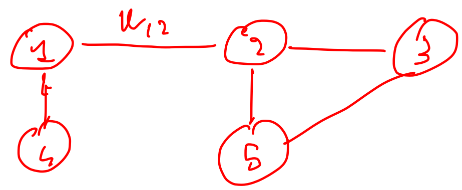
Therefore $w = \begin{bmatrix} 0 & w_2 & \frac{11}{4} w_2 & 0 & 2 w_2 \end{bmatrix}$

From $w \mathbb{1}_5 = 1$ we have $\left(1 + \frac{11}{4} + 2\right) w_2 = 1 \rightarrow w_2 = \frac{4}{23}$

Hence $w = \begin{bmatrix} 0 & \frac{4}{23} & \frac{11}{23} & 0 & \frac{8}{23} \end{bmatrix}$

2. Compute the values of $\alpha$, $\beta$, and $\gamma$ such that the digraph $G$ is the output of the Metropolis-Hasting algorithm.

The underlying undirected graph is




Hence $w_{12} = \frac{1}{\max(d(1), d(2)) + 1} = \frac{1}{4}$

This gives $\beta - \gamma = \frac{1}{4}$. Moreover, in $G$, the weights $w_{23}$ and $w_{32}$ must be identical. Therefore $\alpha = \frac{1}{4}$.

3. Assume that nodes are sensors, each storing a noisy sample $y_i = \theta + v_i$ $i = 1, 2, \dots, 5$ of a common scalar parameter $\theta$. The random variables $v_i$ are independent and Gaussian with zero mean and variance $\sigma_i^2 > 0$. Describe how to use the digraph $G$ obtained in point 2 for computing the BLUE estimate $\hat{\theta}$ of $\theta$ through distributed computations. Provide also the expression of $\hat{\theta}$ .

For each node $i$ run two average consensus algorithms with initial states $x_i(0) = \sigma_i^{-2} y_i$ and $\bar{x}_i(0) = \sigma_i^{-2}$.

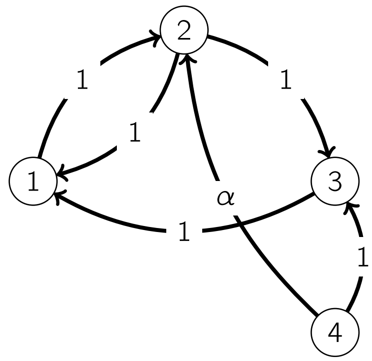
Then, each node computes $\hat{\theta}_i = \frac{x_i(\infty)}{\bar{x}_i(\infty)}$. Indeed, the BLUE estimate can be written as

$$\hat{\theta} = \frac{\frac{1}{5} \sum_{i=1}^{5} \sigma_i^{-2} y_i}{\frac{1}{5} \sum_{i=1}^{n} \sigma_i^{-2}}$$

**Problem 5.**

/15

Consider the Laplacian flow $\dot{x} = -Lx$, $x(0)^T = \begin{bmatrix} 2 & 4 & 4 & 4 \end{bmatrix}$ where $L$ is the Laplacian matrix of the following digraph and $\alpha > 0$.




1. Does $x(t)$ reach consensus as $t \to +\infty$? If yes, compute the consensus value.

• $G$ has the GRNs $\{1, 2, 3\}$. Then $x(t) \to (w^T x(0)) \mathbb{1}_4$ where $w^T L = 0$ and $w^T \mathbb{1}_4 = 1$. We have

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -\alpha & -1 & 1+\alpha \end{bmatrix} \quad \text{and} \quad w^T = \begin{bmatrix} w_1 & w_2 & w_3 & 0 \end{bmatrix}$$

$$w^T L = 0 \to \begin{cases} w_1 - w_2 - w_3 = 0 \to w_2 = w_3 \\ -w_1 + 2w_2 = 0 \to w_1 = 2w_2 \\ -w_2 + w_3 = 0 \to \text{coherent with the first eq.} \\ 0 = 0 \end{cases}$$

Then $w = \begin{bmatrix} 2w_2 & w_2 & w_2 & 0 \end{bmatrix} \xrightarrow{\sum_{i=1}^{4} w_i = 1} (2+1+1) w_2 = 1 \to w_2 = \frac{1}{4}$

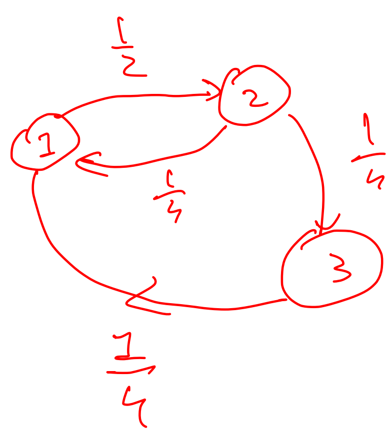
$$w = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

The consensus state is $(w^T x(0)) \mathbb{1}_4 = \frac{1}{2} 2 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 4 = 3$

2. Redefine the weights of the subgraph $\tilde{G}$ spanned by $\{1, 2, 3\}$ in order to obtain average consensus on $\tilde{G}$. Draw $\tilde{G}$.

Average consensus is guaranteed if $\tilde{G}$ is balanced

A possibility is to use the weights $\tilde{a}_{ij} = w_i \, a_{ij}$



**Problem 6.**

/5

Let $G$ be an unweighted strongly connected digraph with binary adjacency matrix $A \in \{0,1\}^{n\times n}$, $n \geq 2$. Let $(\lambda, v)$ be the dominant eigenpair, that is $Av = \lambda v$ and $\mathbb{1}_n^T v = 1$. Consider the matrix $P \in \mathbb{R}^{n\times n}$ with entries

$$P_{ij} = \frac{1}{\lambda}\frac{v_j}{v_i}A_{ij}, \quad i,j \in \{1,2,\dots,n\}$$

1. Show that $v \succ 0$ and that $P$ is row stochastic and irreducible.

- $v > 0$ follows from the Perron-Frobenius theorem since $A$ is irreducible
- $P_{i1} + \dots P_{in} = \frac{1}{\lambda}\frac{v_1}{v_i}A_{i1} + \dots + \frac{1}{\lambda}\frac{v_n}{v_i}A_{in} =$

$$= \frac{1}{\lambda v_i}\begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}\begin{bmatrix} A_{i1} \\ \vdots \\ A_{in} \end{bmatrix} =$$

$$= \frac{1}{\lambda v_i} \underbrace{(A)_{i,\cdot}}_{i,\text{th colum}} v = \frac{1}{\lambda v_i}\lambda v_i = 1$$

↳ hence $P$ stochastic

$P_{ij} \neq 0 \Leftrightarrow A_{ij} \neq 0$. Since $A$ is irreducible, also $P$ is

2. Pick $i,j \in \{1,2,\dots,n\}$ and $k \geq 1$. By considering the weighted digraph associated to $P$, assume there is a path of length $k$ from $i$ to $j$. Show that the product of the edge weights along the path is $\frac{1}{\lambda^k}\frac{v_j}{v_i}$.

the weight of $(i,j)$ in $P$ is $\frac{1}{\lambda}\frac{v_j}{v_i}$

Path : $(i,i_1)(i_1,i_2)\dots(i_{k-1},j)$ → $k$ edges

weight product : $\frac{1}{\lambda}\frac{\cancel{v_{i_1}}}{v_i}\frac{1}{\lambda}\frac{v_{i_2}}{\cancel{v_{i_1}}} \cdot \dots \frac{1}{\lambda}\frac{v_j}{\cancel{v_{i_{k-1}}}} = \frac{1}{\lambda^k}\frac{v_j}{v_i}$